請盡量詳細寫出各解題步驟及計算過程，否則不予計分。

1. (25%) Given a matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 6 \\ 0 & 6 & 3 \end{pmatrix}$,

   (1) (20%) find its eigenvalues and normalized eigenvectors.
   (2) (5%) show that the corresponding eigenvectors are clearly mutually orthogonal.

2. (25%) The differential equation is given by $\dfrac{d^2 f}{dt^2} + 5\dfrac{df}{dt} + 6f = 24e^{-6t}$.

   (1) (18%) Find the general solution of the equation.

   (2) (7%) When $f=0$ and $\dfrac{df}{dt} = 3$ at $t=0$, find the solution of the equation.

3. (15%) Let $f(x) = \begin{cases} x & -\pi < x < 0 \\ 0 & 0 \le x < \pi \end{cases}$, expand $f(x)$ as a Fourier Series.

   Hint: $\int x\cos(nx)dx = \dfrac{1}{\pi n^2}\left[\cos(nx) + nx\sin(nx)\right]$, $\int x\sin(nx)dx = \dfrac{1}{\pi n^2}\left[-nx\cos(nx) + \sin(nx)\right]$

4. (35%) Two vector fields are given by $\mathbf{a}=(x+y)\mathbf{i}+(y-x)\mathbf{j}+(y+z^2)\mathbf{k}$ and $\mathbf{b}=(xy^2+z)\mathbf{i}+(x^2y+2)\mathbf{j}+x\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in Cartesian coordinate system, A and B are the points (1, 2, 4) and (3, 6, 36).

   (a) (5%) Calculate $\nabla\cdot\mathbf{a}$, $\nabla\cdot\mathbf{b}$, $\nabla\times\mathbf{a}$ and $\nabla\times\mathbf{b}$.
   (b) (5%) Which one ($\mathbf{a}$ and $\mathbf{b}$) is a conservative field? Why?

   (c) (15%) Evaluate directly the line integrals $I = \int_A^B \mathbf{b}\cdot d\mathbf{r}$ along the following two paths

   (i) C_1 given by $x=t$, $y=2t$ and $z=4t^2$. Hint: at A, $t_A=1$ and at B, $t_B=3$.
   (ii) C_2 given by $x=1+0.5t^2$, $y=2+2t$ and $z=4+16t$. Hint: at A, $t_A=0$ and at B, $t_B=2$.

   (d) (10%) Find the potential function of the conservative field, i.e., find a scalar function ϕ to make $\mathbf{a}$ or $\mathbf{b}=\nabla\phi$.