淡江大學 106 學年度日間部寒假轉學生招生考試試題

系別：物理學系三年級　　　科目：應用數學　　　28-1

考試日期：1 月 6 日 (星期六) 第 2 節　　　本試題共 5 大題，1 頁

Each problem is worth 20 points; parts of problems have equal value.

1. (20%) Consider the differential equation
$$xy'' - (x-1)y' - y = 0.$$
   (a) Verify that $y_1(x) = e^x$ is a solution.
   (b) Find the Wronskian for the equation.
   (c) Use variation of constant to find the second solution.
   (d) Find the general solution of the equation.

2. (20%) Use the Frobenius method to solve the equation
$$2xy'' + y' + y = 0,$$
i.e., find the solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ with $a_0 \neq 0$.
   (a) Show that $x = 0$ is a regular singular point of the equation.
   (b) Find the possible values of s.
   (c) Obtain the recursion relation with respect to each value of s.
   (d) Find the general solution by giving the first four nonzero terms in each series.

3. (20%) Let $f(x) = x$ for $0 < x < 1$.
   (a) Expand $f(x)$ in a Fourier cosine series.
   (b) Sketch the Fourier cosine series over three periods on a symmetric interval.
   (c) Describe the convergence of the Fourier cosine series of $f(x)$ found in (b).
   (d) Usie the Parseval identity to find the value of an infinite series.

4. (20%) Consider the eigenvalue problem
$$y'' + \lambda y = 0 \quad (0 < x < 1).$$
   (a) Solve the problem with boundary conditions $y(0) = 0$ and $y(1) = 0$.
   (b) Continued from (a), determine if $\lambda = 0$ is an eigenvalue.
   (c) Solve the problem with boundary conditions $y'(0) = 0$ and $y'(1) = 0$.
   (d) Continued from (c), determine if $\lambda = 0$ is an eigenvalue.

5. (20%) The first three Legendre polynomials are
$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1),$$
where $-1 < x < 1$.
   (a) Calculate $P_3(x)$ from the Rodrigues formula.
   (b) Verify that $P_2(x)$ and $P_3(x)$ are orthogonal.
   (c) Express the polynomial $f(x) = 5x^3 - 3x^2 - x + 1$ as a linear combination of the Legendre polynomials.
   (d) Expand $g(x) = |x|$ in a Fourier–Legendre series on $-1 < x < 1$. Explicitly calculate the Fourier–Legendre coefficients a_0, a_1, a_2, and a_3.