淡江大學106學年度碩士班招生考試試題

35-1

系別：數學學系A組　　　　　　科目：線性代數

考試日期：3月4日(星期六) 第2節　　　　　　本試題共　6　大題，　2　頁

本試題雙面印刷

1. (20 pts) True or False. Explain the reason for your answer briefly.

   (a) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$. Then A is diagonalizable.

   (b) Let P be an invertible 3×3 matrix over $\mathbb{R}$. Then $P \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} P^{-1}$ is invertible.

   (c) Let $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \right\}$. Then β is a basis for $\mathbb{R}^3$.

   (d) Let A and B be two 3×3 matrices. Suppose A and B have the same characteristic polynomial. Then A is diagonalizable if and only if B is diagonalizable.

2. (10 pts) We can define an inner product on $\mathbb{R}^3$ by

$$< x, y >= \sum_{i=1}^{3} x_i y_i, \text{ for all } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

   Use Gram-Schmidt process to construct an orthonormal basis γ from the basis

$$\beta = \{(-1,0,1)^t,\ (1,1,0)^t,\ (1,1,-1)^t\}.$$

3. Let $A = \begin{bmatrix} \frac{3}{4} & 0 & -\frac{5}{4} \\ \frac{3}{2} & -1 & -\frac{3}{2} \\ -\frac{5}{4} & 0 & \frac{3}{4} \end{bmatrix}$. Define $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$L_A(v) = Av, \text{ for all } v \in \mathbb{R}^3.$$

   (a) (10pts) Find all eigenvalues and eigenvectors of L_A.

   (b) (5pt s) Find a matrix P so that $D = P^{-1}AP$ is diagonal matrix. What is D ?

4. (15 pts) Let $B = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 3/2 & -1/2 \\ 2 & -1/2 & -1/2 \end{bmatrix}$. Find matrices Q and J so that $J = Q^{-1}BQ$ is the Jordan canonical form for B.

5. (20 pts) Denote V a vector space of finite dimension over $\mathbb{R}$ and $T : V \to V$ a linear transformation. Prove the following statements are equivalent.

   (a) $T(v) = 0$ for $v \in V$ if and only if $v = 0$.

   (b) T is 1-1.

   (c) $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for V.

6. (20 pts) Let A be an $n \times n$ real symmetric matrix. Show that $\max\limits_{x \in \mathbb{R}^n, x \neq 0} \dfrac{< Ax, x >}{< x, x >}$ is the largest eigenvalue of A and $\min\limits_{x \in \mathbb{R}^n, x \neq 0} \dfrac{< Ax, x >}{< x, x >}$ is the smallest eigenvalue of A