淡江大學 101 學年度碩士班招生考試試題

系別：數學學系　　　　科目：高等微積分

考試日期：2 月 26 日(星期日) 第 2 節　　　　本試題共 7 大題， 1 頁

1. (10 points) Test the following series for convergence

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{e^{-k}}{\sqrt{k+1}}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{3^k}$

2. (15 points) Show that $f(x) = \dfrac{x}{1+x^2}$ is uniformly continuous on $\mathbb{R}$.

3. (15 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$$

   if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ exist at $(0,0)$ but not equal.

4. (15 points) Let $f_n : [1,2] \to \mathbb{R}$ be defined by $f_n(x) = x/(1+x)^n$.

   (a) Prove that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for $x \in [1,2]$.

   (b) Is it uniformly convergent ?

   (c) Is $\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)\, dx$?

5. (15 points) A real-valued function defined on (a,b) is called convex when the following inequality holds for $x, y \in (a,b)$ and $t \in (0,1)$:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

   If f has a continuous second derivative and $f'' > 0$, show that f is convex.

6. (15 points) Prove that

$$\log 2 = \lim_{n\to\infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}].$$

7. (15 points)

   (a) Define $x : \mathbb{R}^2 \to \mathbb{R}$ by $x(r,\theta) = r\cos\theta$ and $y : \mathbb{R}^2 \to \mathbb{R}$ by $y(r,\theta) = r\sin\theta$. Show that

$$\frac{\partial(x,y)}{\partial(r,\theta)}(r_0,\theta_0) = r_0.$$

   (b) Show that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$