淡江大學99學年度碩士班招生考試試題

系別：數學學系　　　　　　　　科目：高 等 微 積 分

准帶項目請打「V」	
	計 算 機

本試題共 1 頁，7 大題

1. Let X, Y be metric spaces, and $f: X \to Y$ be continuous.
   (a) Show that if X is compact, then $f(X)$ is compact.
   (b) Show that if $X = [a, b]$, and $Y = \mathbb{R}$, then f has a maximum and a minimum. (15%)

2. Let $f: (-1, 1) \to \mathbb{R}$ be continuous. Assume that f is differentiable on $(-1, 0) \cup (0, 1)$ and $\lim_{x \to 0} f'(x) = C$.
   Show that f is differentiable at 0 and $f'(0) = C$. (10%).

3. Let $a_0 = 1$, $a_1 = 1$ and $a_{n+2} = a_n + a_{n+1}$ for $n \geq 0$.
   (a) Let $C_n = \frac{a_{n+1}}{a_n}$ for all n, show that $\{C_n\}$ converges.
   (b) Calculate $\lim_{n \to \infty} C_n$
   (Hint. Consider $\{C_{2n}\}$ and $\{C_{2n+1}\}$ separately.) (15%).

4. For $n \in \mathbb{N}$, let $f_n: [0, 1] \to \mathbb{R}$ be continuous.
   If $\{f_n\}$ converges uniformly to a function f, show that f is continuous. (15%)

5. Let $f: [0, 1] \to \mathbb{R}$ be Riemann integrable and $\int_0^1 f(x)\,dx = 0$
   (a) Show by an example that not all $f(x)$'s are necessarily 0.
   (b) If f is continuous, show that $f(x) = 0$ for any x in $[0, 1]$.
   (15%)

6. Let $f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$
   (a) Find the directional derivatives of f at $(0, 0)$.
   (b) Is f differentiable at $(0, 0)$? (15%).

7. Find the maximum and the minimum of $f(x, y, z) = x + y + z$ subject to the conditions $x^2 + y^2 = 2$ and $x + z = 1$.
   (15%).