

## 淡江大學 103 學年度碩士班招生考試試題

系別：數學學系

科目：基礎代數（含線性代數、代數學）

考試日期：3月2日(星期日) 第3節

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1. (15 pts) Let  $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ -1 & -3 & 3 & 0 \\ 2 & 6 & 9 & 5 \end{bmatrix}$ .

- (a) Find a basis for the column space of  $A$ .  
 (b) Find a basis for the null space of  $A$ .

2. (15 pts) Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear operator defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_3, -2x_1 + x_2, x_1 + 2x_2 + 3x_3).$$

- (a) Find the matrix representation of  $T$  with respect to the basis  $\{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$ .  
 (b) Show that  $T$  is invertible and find its inverse  $T^{-1}(x_1, x_2, x_3)$ .

3. (15 pts) (a) Show that  $f(x) = x^5 + x^2 + 1$  is irreducible in  $\mathbb{Z}_2[x]$ .  
 (b) Show that  $g(x) = x^5 + 2x^4 + 2x^3 + 3x^2 + 2x + 1$  is irreducible in  $\mathbb{Q}[x]$ .

4. (15 pts) Prove that every subgroup of a finite cyclic group is cyclic.

5. (10 pts) Let  $G$  be a group with identity element  $e$ . Let  $p$  be a prime. If  $H$  and  $K$  are both subgroups of order  $p$  in  $G$ , show that either  $H = K$  or  $H \cap K = \{e\}$ .

6. (10 pts) Let  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ .

- (a) Find the eigenvalues of  $A$ .  
 (b) Find an orthogonal matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ .

7. (10 pts) Let  $\phi$  be a ring homomorphism from a ring  $R$  into a field  $F$ . Prove that  $\text{Ker}(\phi)$  is a prime ideal in  $R$ .

8. (10 pts) Suppose  $A$  is a  $7 \times 5$  matrix and  $B$  is a  $5 \times 7$  matrix. Show that  $\det(AB) = 0$ .