

## 淡江大學 101 學年度碩士班招生考試試題

系別：數學學系

科目：線性代數 50% 及 代數學 50%

考試日期：2月26日(星期日) 第3節

本試題共 8

大題， 2

頁 P.1.

1. (15 points) Let  $A = \begin{bmatrix} 1 & 3 & -1 & 4 & 1 \\ 2 & 1 & 8 & 3 & 0 \\ -1 & 2 & -9 & 1 & 1 \\ 1 & 4 & -3 & 5 & 0 \end{bmatrix}$

- (a) Find a basis for the row space of  $A$ .
- (b) Find a basis for the null space of  $A$ .
- (b) Let  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$  be given by  $T(X) = AX$  for column vectors  $X \in \mathbb{R}^5$ . Find a basis for the image of  $T$ .
2. (15 points) Let  $P_2$  be the space of polynomials of degree  $\leq 2$  over  $\mathbb{R}$ . Let  $T : P_2 \rightarrow P_2$  be given by  $(Tf)(x) = xf'(x) - f(x+1)$ .
- (a) Find the matrix representation of  $T$  with respect to the basis  $\{1, x, x^2\}$ .
- (b) Find the eigenvalues and eigenvectors of  $T$  and determine whether  $T$  is diagonalizable.
3. (10 points) Let  $V = \{(x, y, z, w) \mid x - 2y + 4z + 3w = 0, 2x - y - z = 0\} \subset \mathbb{R}^4$ .
- (a) Find an orthogonal basis for  $V$ .
- (b) Let  $X = (1, -2, 1, 6)$  be a vector in  $\mathbb{R}^4$ . Find the orthogonal projection of  $X$  on  $W$ .
4. (10 points) Let  $T$  be a linear operator on a finite dimensional vector space  $V$ . Suppose  $T$  is idempotent, that is  $T^2 = T$ . Prove that
- (a)  $V = \ker(T) \oplus \text{range}(T)$ .
- (b)  $T$  is diagonalizable.
5. (10 points)
- (a) Let  $G = \langle a \rangle$  be a cyclic group of order 30. Find the order and the index of the subgroup  $\langle a^8 \rangle$ .
- (b) Let  $G$  be a group which has subgroups of order 12 and 40. What is the minimum possible order of  $G$ ?
6. (10 points) Let  $G = GL_n(\mathbb{R})$  be the group of nonsingular  $n \times n$  real matrices and  $H = SL_n(\mathbb{R})$  be the subgroup of  $G$  consisting of  $n \times n$  real matrices of determinant 1. Show that  $G/H$  is isomorphic to  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ .
7. (15 points) Let  $f(x) = x^4 - x^2 + 1$ .
- (a) Show that  $f(x)$  is irreducible over  $\mathbb{Q}$ .
- (b) Construct a field which contains a root of  $f(x)$ .

# 淡江大學 101 學年度碩士班招生考試試題

14-2

系別：數學學系

科目：線性代數 50% 及 代數學 50%

考試日期：2月26日(星期日) 第3節

本試題共 8 大題， 2 頁 p.2.

8. (15 points) Let  $p$  be a prime integer and let  $T = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, (a, b) = 1, p \nmid b \right\}$  and  $I = \left\{ \frac{a}{b} \in T \mid p \mid a \right\}$ .
- (a) Prove that  $I$  is an ideal of  $T$ .
- (b) Show that  $T/I \simeq \mathbb{Z}_p$ .