淡江大學八十七學年度碩士班入學考試試題

系別： 數學系　　　科目： 線性代數

1. Let A be an $m \times n$ matrix over real numbers $\mathbf{R}$.

(a) (5 points) Prove or disprove: $AX = 0$ has a non-trivial solution implies that $A^t X = 0$ has a non-trivial solution.

(b) (5 points) Prove or disprove: $AX = 0$ has a non-trivial solution if and only if $rank A < n$.

(c) (5 points) Prove or disprove: $AX = 0$ if and only if $A^t AX = 0$

2. Let V be the vector space of polynomials over the complex numbers $\mathbf{C}$ which are of degree ≤ 3. Let $T : V \longrightarrow V$ be the linear transformation defined by $T(f) = f + xf''$.

(a) (5 points) Find the matrix A which represents T with respect to the basis $\{1, x, x^2, x^3\}$.

(b) (5 points) Find the Jordan form for A.

(c) (10 points) Find a basis in V such that the matrix representation of T is the Jordan form found in (b).

3. Let V be the space of $n \times 1$ matrices over field $\mathbf{F}$. $D : V^2 \longrightarrow \mathbf{F}$ is a bilinear form satisfying $D(\alpha, \alpha) = 0$ for all $\alpha \in V$.

(a) (5 points) Show that there is a constant c such that $D(\alpha, \beta) = c\det(A)$ where A is the matrix with column vectors α, β.

(b) (5 points) Let W be the space of bilinear forms on V. Determine diemnsion of W.

4. Let V be a finite dimensional inner product space over complex number $\mathbf{C}$. Let T be a linear operator on V and T^* be its adjoint operator.

(a) (5 points) Let $\mathcal{B}$ be an orthonormal basis for V and A be the matrix representation of T with respect to $\mathcal{B}$. Show that the matrix representation of T^* with respect to $\mathcal{B}$ is A^*.

(b) (5 points) Show that $range(T^*) = ker(T)^{\perp}$.

5. Let V be a finite dimensional inner product space over the complex numbers $\mathbf{C}$ with inner product denoted by $\langle \quad , \quad \rangle$ and let T be a linear operator on V.

(a) (10 points) Show that T is self-adjoint if and only if $\langle T(\alpha), \alpha \rangle$ is real. Deduce that all characteristic values for T are real.

(b) (5 points) Show that if T is normal, then characteristic vectors corresponding to distinct characteristic values are orthogonal.

6. Let

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

(a) (5 points) Define $\langle X, Y \rangle = Y^t AX$. Show that this defines an inner product on the space V of 3×1 matrices over real numbers $\mathbf{R}$ and write the explicit fromula for this inner product.

(b) (5 points) Find a real orthogonal matrix P such that $P^t AP$ is a diagonal matrix D.

(c) (5 points) Give explicitly an inner product preserving isomorphism between the space V with the above defined inner product and $\mathbf{R}^3$ with the standard inner product.

7. Let V be a vector space of dimension n over a field $\mathbf{F}$. Let E be a projection of V onto a subspace W of dimension m.

(a) (5 points)Show that E is diagonalizable. Find all characteristic values of E indicating its multiplicity.

(b) (5 points) If V is an inner product space and E is an orthogonal projection, show that E is self-adjoint.

(c) (5 points) If $\mathcal{B} = \{\alpha_1, \ldots, \alpha_m\}$ be an orthonormal basis of W and β be a vector in V. Find explicitly the point in W which is closest to β.