24-1

淡江大學96學年度碩士班招生考試試題

系別：數學學系　　　　科目：代　數　學

准帶項目請打「V」	
	簡單型計算機

本試題共 1 頁

1. (a) (5 points) Let m, n and r be integers. If m and r are relatively prime, show that $r \mid mn$ implies that $r \mid n$.

(b) (5 points) Show that $p \in \mathbb{Z}$ is a prime if and only for all integers m and n, $p \mid mn$ implies $p \mid m$ or $p \mid n$.

2. Let G be a group of order $|G| = 2006 = 2 \times 17 \times 59$.

(a) (5 points) Show that G has a normal subgroup of index 2.

(b) (10 points) Show that G is either isomorphic to a cyclic group or a dihedral group.

3. Let $R = \mathbb{Z}[\sqrt{-1}] = \{a + b\sqrt{-1} \mid a, b \in \mathbb{Z}\}$.

(a) (10 points) Show that if $a^2 + b^2$ is a prime in $\mathbb{Z}$ then $a + b\sqrt{-1}$ is a prime in R. Give an example to show that the converse is not true.

(b) (5 points) Let $I = < 1 + 3\sqrt{-1} >$ be the ideal generated by $1 + 3\sqrt{-1}$. Show that $\mathbb{Z}[\sqrt{-1}]/I \cong \mathbb{Z}_{10}$.

(c) (5 points) Show that in general if $I \subset R$ is an ideal, then R/I is finite.

4. Let G be a group of order pq where $p < q$ are primes.

(a) (5 points) Show that G has a normal q-Sylow group.

(b) (5 points) Show that G is cyclic if $p \nmid q - 1$.

(c) (10 points) Find all groups of order 21 up to isomorphism.

5. Let $a, b_1, \cdots, b_{m-2} \in \mathbb{Z}$ such that $a > 0$. Let $p \in \mathbb{Z}$ be a prime. Let

$$g_n(t) = (t^2 + a)(t - b_1) \cdots (t - b_{m-2}) + \frac{p}{pn}$$

(a) (10 points) Show that g_n is irreducible over $\mathbb{Q}$.

(b) (10 points) Show that g_n has $m - 2$ real roots and a pair of complex roots.

6. Let R be a principal ideal domain.

(a) (10 points) Let $(a_1) \subseteq (a_2) \subseteq \cdots\cdots$ be a chain of ideals in R. Show that there is $m \in \mathbb{N}$ such that $(a_r) = (a_s)$ for $r, s \geq m$.

(b) (5 points) Show that for any $a \in R$ there is an irreducible number p such that $p \mid a$.