

淡江大學九十二學年度碩士班招生考試試題

系別：數學學系

科目：代 數 學

准帶項目請打「○」否則打「×」

簡單型計算機

X

本試題共 1 頁

1. Let S be any subset of a group G . Show that $H_S = \{x \in G \mid xs = sx \text{ for all } s \in S\}$ is a subgroup of G . (10%)
2. Show that a group homomorphism $f : G \rightarrow G'$ is a one-to-one map if and only if $\text{Ker}(f) = \{e\}$, where e is the identity of G . (10%)
3. Let $f : G \rightarrow G'$ be a group homomorphism, and let N' be a normal subgroup of G' . Show that $f^{-1}(N')$ is normal subgroup of G . (10%)
4. Let $f : R \rightarrow R'$ be a ring homomorphism and let N be an ideal of R . Show that $f(N)$ is an ideal of $f(R)$. (10%)
5. Show that (x) is a maximal ideal of $\mathbb{Q}[x]$, where $\mathbb{Q}[x]$ is the ring of polynomials over the field of rational numbers \mathbb{Q} . (10%)
6. Let $n \in \mathbb{Z}^+$ be square free, that is, not divisible by the square of any prime integer. Let $\mathbb{Z}[\sqrt{-n}] = \{a + ib\sqrt{-n} \mid a, b \in \mathbb{Z}\}$.
 - (a) Show that the norm N , defined by $N\alpha = a^2 + nb^2$ for $\alpha = a + ib\sqrt{-n}$, is a multiplicative norm on $\mathbb{Z}[\sqrt{-n}]$.
 - (b) Show that $N\alpha = 1$ if and only if α is a unit of $\mathbb{Z}[\sqrt{-n}]$. (20%)
7. Let E be a finite extension of a field F , and let $p(x) \in F[x]$ be irreducible over F and have a degree that is not a divisor of $[E : F]$. Show that $p(x)$ has no zero in E . (10%)
8. Show that the cyclotomic polynomial

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$
 is irreducible over \mathbb{Q} for every prime p . (10%)
9. (a) Describe the elements of the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$.
 (b) To what group is $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$ isomorphic? (10%)