淡江大學九十學年度碩士班招生考試試題

系別：數學學系

科目：代　數　學

准帶項目請打「○」否則打「X」	
計算機	字典
X	X

本試題共 1 頁

1. Let G be a cyclic group of order n, and let d be a positive divisor of n.
(a) Prove that G has a unique subgroup H of order d. (5%)
(b) Prove that G has a unique subgroup K which is isomorphic the quotient group G/H. (10%)

2. Let K be a Sylow p-subgroup of G and N a normal subgroup of G. Suppose that K is normal in N. Show that K is also normal in N. (10%)

3. Let R be a commutative ring with identity and

$$SL_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R,\ ad - bc = 1 \right\},$$
$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R) : c = 0 \right\}.$$

(a) Show that $SL_2(R)$ is a group under usual matrix multiplication. (5%)
(b) Prove or disprove that B is a subgroup of $SL_2(R)$. (5%)
(c) Prove or disprove that B is a normal subgroup of $SL_2(R)$. (5%)

4. Let I, J be ideals of R, and let $I + J = \{a + b : a \in I,\ b \in J\}$.
(a) Prove that both $I \cap J$ and $I + J$ are also ideals of R. (8%)
(b) Consider $R = \mathbb{Z}$, the ring of integers, and I, J ideals generated by integers m, n respectively. Find generators for the ideals $I \cap J$ and $I + J$. (7%)

5. Prive that the ring $\mathbb{Z}[\sqrt{-13}] = \{a + b\sqrt{-13} : a, b \in \mathbb{Z}\}$ is not a principal ideal domain. (Hint: consider the element $1 + \sqrt{-13}$.) (10%)

6. Prove that each polynomial is irreducible over the rational numbers. (15%)
(a) $x^3 + 2x + 1$, (b) $x^6 - 6x^5 + 4x^3 + 7$, (c) $16x^3 + 3x^2 + 5x + 10$.
(Hint: Use the root test, Eisenatein's criterion, or consider modulo a suitable prime p.)

7. Let K be the splitting field of the polynomial $x^4 + 2x^3 + 2x - 1$ over $\mathbb{Q}$.
(a) Determine the extension degree $[K : \mathbb{Q}]$. (8%)
(b) Find the Galois group of K over $\mathbb{Q}$. (6%)
(c) Find all intermediate fields between $\mathbb{Q}$ and K. (6%)