淡江大學八十八學年度碩士班招生考試試題

系別：數學學系　　　科目：代　數　學　　　本試題共 1 頁

Answer all questions . Show all work.

1. Let $f : G \to H$ be a surjective homomorphism of groups with kernel K and let M be a subgroup of H.
   (a) Prove that there is a subgroup N of G such that $K \subseteq N \subseteq G$ and N/K is isomorphic to M.
   (b) If M is normal in H, prove that N is normal in G and G/N is isomorphic to H/M.
   (15%)

2. Let M and N be normal subgroups of a group G with $M \cap N = < e >$. Prove that G is isomorphic to a subgroup of $G/M \times G/N$. (11%)

3. Let K be a Sylow p-subgroup of G and N a normal subgroup of G.If K is a normal subgroup of N, prove that K is normal in G. (11%)

4. Show that the principal ideal (x) in $Z[x]$ is prime but not maximal. (14%)

5. Show that $x^2 - 3$ and $x^2 - 2x - 2$ are irreducible in $Q[x]$. Find their splitting fields. (11%)

6. (a) Let P be the ideal $\{2a + (1 + \sqrt{-5})b \mid a, b \in Z[\sqrt{-5}]\}$ in $Z[\sqrt{-5}]$. Prove that $r + s\sqrt{-5} \in P$ if and only if $r \equiv s \ (mod\ 2)$.
   (b) Show that P^2 is the principal ideal (2).
   (15%)

7. (a) Let $\omega = \frac{-1+\sqrt{3}i}{2}$ be a complex root of 1. Find the minimal polynomial $p(x)$ of ω over Q and show that ω^2 is also a root of $p(x)$.
   (b) Find the Galois group $Gal_Q Q(\omega)$.
   (11%)

8. (a) Prove that the set S of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b, c \in$ R is a subring of $M(R)$, where $M(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in R\}$.
   (b) Prove that the set I of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $b \in R$ is an ideal in the ring S.
   (c) Show that there are infinitely many distinct cosets in S/I, one for each pair in $R \times R$.
   (12%)