系別：電機工程學系   科目：工　程　數　學

1. Linear Algebra

True or False. If you think the statement is true, mark "T"; if you think the statement is false, mark "F". You don't need to write down your reasons.

(a) (5%) Every real 3×3 matrix having $\lambda = 1 + i$ as an eigenvalue is diagonalizable over the complex numbers. (Note: $i = \sqrt{-1}$)

(b) (5%) If matrix A is similar to matrix B, then $\text{rank}(A) = \text{rank}(B)$.

(c) (5%) If A is an $m \times n$ matrix having orthonormal columns, then $\det(AA^T) = \det(A^T A)$.

(d) (5%) If A is a symmetric matrix such that $A^5 = 0$, then $A = 0$.

(e) (5%) If A and B are 2×2 matrices with the same trace and the same determinant, then A and B have the same eignevalues.

2. Linear Algebra

Suppose the singular value decomposition of a matrix A is $A = U\Sigma V^T$ where

$$U = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) (5%) Find the eigenvalues of $A^T A$.

(b) (5%) Find a basis for the nullspace of A.

(c) (5%) Find a basis for the column space of A.

3. Complex Variables

(a) (10%) Find the residue at $z = 0$ of $z^{-3}\csc(z^2)$.

(b) (10%) Use residual theorem to compute the integral $\int_0^\infty \frac{1}{x^4 + a^4}dx, \quad a > 0$.

4. Differential Equations

Consider a dynamical system described by the following set of two first-order differential equations

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) (10%) Use eigenvalue/eigenvector concept to find $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

(b) (10%) Write down the equivalent second-order differential equation and the initial conditions in terms of $x_1(t)$ for the dynamical system. Use "Laplace transform" to solve the equivalent second-order differential equation and determine $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

5. Fourier Series

(a) (10%) Find the Fourier series of $|\sin(x/2)|$ on $[0, 2\pi]$.

(b) (10%) Use the result in (a) to find $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$.