請詳列計算過程，否則不予計分，每題 10 分，共 100 分

1. Find the a unit vector (單位向量) that is oppositively directed (相反方向) to the vector $\left(-3,1,\sqrt{6},3\right)$.

2. Show that the vectors $\mathbf{v}_1 = (1,2,1)$, $\mathbf{v}_2 = (2,9,0)$, and $\mathbf{v}_3 = (3,3,4)$ form a basis for $R^3$.

3. Show that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible for all value of $\theta$.

4. $A$ is a $4 \times 4$ matrix and its determinat is $\det(A) = -2$, find $\det(-A)$, $\det(A^{-1})$, $\det(2A^T)$, $\det(A^3)$, and $\det\left((3A)^{-1}\right)$

5. Solve by Gauss–Jordan elimination.

$$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &=& 0 \\ 2x_1 + 6x_2 - 5x_3 - \ 2x_4 + 4x_5 - \ 3x_6 &=& -1 \\ 5x_3 + 10x_4 \qquad\quad + 15x_6 &=& 5 \\ 2x_1 + 6x_2 \qquad\quad + \ 8x_4 + 4x_5 + 18x_6 &=& 6 \end{array}$$

6. Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 4 & 2 & 0 \\ -1 & -3 & 0 & 5 \end{bmatrix}$.

7. Find the eigenvalues of the upper triangular matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$.

8. Find bases for the eigenspaces of the matrix $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

9. Find a matrix $P$ that diagonalizes $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

10. Show that $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}$ are not similar matrices.