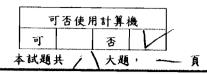
系別:數學學系二年級

科目:線性代數



Partial credit-You must show all your work.

1. (10 %) Consider the system of equations

$$2x_1 + 5x_2 - x_3 = a$$
$$x_1 + 2x_2 = b$$
$$3x_1 + 7x_2 - x_3 = c$$

Determine conditions on a, b, c that are necessary and sufficient for the system to be consistent.

2. (10 %) Let  $S_1$  and  $S_2$  be subspaces of a vector space V. Suppose  $S_1 \neq \{0\}$ ,  $S_1 \neq V$ ,  $S_2 \neq \{0\}$  and  $S_2 \neq V$ . Show that there exists a vector  $\mathbf{v} \in V$  such that  $\mathbf{v} \notin S_1$  and  $\mathbf{v} \notin S_2$ .

3. (10 %) Let 
$$A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 3 \end{bmatrix}$$
. Find  $A^{-1}$ .

4. Let  $M_{33}=\left\{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\middle|a_{ij}\in\mathbb{R}\right\}$  be the vector space of all  $3\times 3$  matrices. Define a linear transformation  $T:M_{33}\to\mathbb{R}$  by

$$T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11} + a_{22} + a_{33}.$$

- (1) (10 %) Find a basis for ker T.
- (2) (10 %) Find nullity T and rank T.

5. Let  $T: \mathbb{R}^3 \to \mathbb{R}^3$  be a linear transformation given by

$$T(a, b, c) = (2a + b, b - c, c - 3a).$$

Let  $\mathcal{B}_1 = \{(1,1,0), (1,0,1), (0,1,0)\}$  and  $\mathcal{B}_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$  be two bases of  $\mathbb{R}^3$ .

- (1) (10 %) Find  $[T]_{\mathcal{B}_1}$  (the matrix of T with respect to the basis  $\mathcal{B}_1$ ) and  $[T]_{\mathcal{B}_2}$  (the matrix of T with respect to the basis  $\mathcal{B}_2$ ).
- (2) (10 %) Find a matrix P such that  $P^{-1}[T]_{B_1}P = [T]_{B_2}$ .

6. (10%) Let A be a  $3 \times 3$  real orthogonal matrix. Suppose  $\lambda \in \mathbb{R}$  is an eigenvalue of A. Show that either  $\lambda = 1$  or  $\lambda = -1$ .

7. (10 %) Let  $T: V \to W$  be a one-to-one linear transformation with dim  $V = n = \dim W$ . Show that T is onto.

8. (10 %) Let  $A = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$ . Find an invertible matrix P that makes  $P^{-1}AP$  a diagonal matrix.