Please show your work.

**1. (20 pts)** (a) Suppose $p$ is a prime number and $a$ is an integer, $(a, p) = 1$. Prove that $a^{p-1} \equiv 1 \pmod{p}$.

(b) What is the remainder when $35^{35}$ is divided by 37?

**2. (12 pts)** Prove or disprove: If $G$ is a group of order 53, then $G$ must be cyclic.

**3. (12 pts)** Suppose $G = \{e, a, b, c\}$ is a group of order 4; but it contains no element of order 4. Write out the operation table for $G$.

**4. (24 pts)** (a) Prove that every finite integral domain is a field.

(b) Give an example of an integral domain which is not a field.

**5. (12 pts)** Show that the principal ideal $\langle x - 1 \rangle$ in $\mathbb{Z}[x]$ is prime but not maximal.

**6. (20 pts)** (a) Show that $x^3 + x + 1$ is irreducible in $\mathbb{Z}_5[x]$.

(b) Let $R$ be the quotient ring $\mathbb{Z}_5[x]/\langle x^3 + x + 1 \rangle$. How many elements are there in $R$? Is $R$ a field? Please justify your answer.