淡江大學98學年度轉學生招生考試試題

系別：數學學系數學組三年級　　　　科目：代　　數

准帶項目請打「V」	
	計 算 機

本試題共 6 大題， 頁

1. (a) (10 points) Let $a,\ b \in \mathbb{Z}$ and $d = \gcd\{a,\ b\}$. Show that there exists $r\ s \in \mathbb{Z}$ such that $ra + sb = d$.

(b) (5 points) Let a, b and c be integers. If a and c are relatively prime, show that $c \mid ab$ implies that $c \mid b$.

(c) (5 points) Show that $p \in \mathbb{Z}$ is a prime if and only for all integers a and b, $p \mid ab$ implies $p \mid a$ or $p \mid b$.

2. (15 points) Let $R = \mathbb{Z}[\sqrt{-1}] = \{a + b\sqrt{-1} \mid a,\ b \in \mathbb{Z}\}$.

(a) Show that if $a^2 + b^2$ is a prime in $\mathbb{Z}$ then $a + b\sqrt{-1}$ is a prime in R. Give an example to show that the converse is not true.

(b) Let $\alpha = 9 + 7\sqrt{-1}$ and $\beta = 1 - 4\sqrt{-1}$. Find $q,\ r \in R$ such that $\alpha = q\beta + r$ such that $|r| < |\beta|$.

(c) Show that if $I \subset R$ is a non-zero ideal, then R/I is finite.

3. (15 points)

(a) Construct a field F over $\mathbb{Q}$ such that $x^7 + 2x + 2$ has a root in F. Find the degree of extension of F over $\mathbb{Q}$.

(b) Construct a finite field of 27 elements.

4. (20 points) Let $G = \langle a \rangle$ be a cyclic group of order n.

(a) Show that for any $d \mid n$, there is a subgroup of order d.

(b) Show that if $(r, n) = d$, then $a^d \in \langle a^r \rangle$.

(c) Show that the subgroup $\langle a^r \rangle$ has order $\frac{n}{(r,n)}$.

5. (15 points)

(a) Explain why non-trivial group homomorphism $\phi : \mathbb{Z}_8 \longrightarrow \mathbb{Z}_{15}$ does not exist.

(b) Show that a non-trivial group homomorphism $\phi : \mathbb{Z}_{15} \longrightarrow \mathbb{Z}_9$ exist by exhibiting an example.

6. (15 points) Let R be an integral domain.

(a) Show that every prime element in R is irreducible.

(b) Suppose that R is a PID. Show that every irreducible element in R is a prime.