淡江大學九十四學年度轉學生招生考試試題

系別： 數學學系三年級　　科目：線 性 代 數

准帶項目請打「V」	
	簡單型計算機

節次： 7 月 13 日第 5 節
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1. (5 points) Let A and B be $n \times n$ matrices. Is

$$(A+B)^2 = A^2 + 2AB + B^2?$$

If so, prove it; if not, give a counterexample(反例) and state under what conditions the equation is true.

2. (5 points) Let a, b and c be real numbers(實數) such that $abc \neq 0$. Prove that the plane $ax + by + cz = 0$ is a subspace(子空間) of $\mathbb{R}^3$.

3. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If $T([1,0,0]) = [-3,1]$, $T([0,1,0]) = [4,-1]$, and $T([0,-1,1]) = [3,5]$, find $T([-1,4,2])$.

4. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}$.
   (a) (5 points) Find the standard matrix representation(標準矩陣表示式) of L.
   (b) (5 points) Show that L is invertible(可逆).
   (c) (5 points) Find a formula for L^{-1}.

5. (10 points) Let V be a vector space with basis(基底) $\{v_1, v_2, v_3\}$. Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also a basis for V.

6. Given a linear system,

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -2 \\ 2x_1 & + & x_2 & + & x_3 & = & 0 \\ 3x_1 & - & x_2 & + & 5x_3 & = & 1 \end{array}$$

   (a) (5 points) Transfer this system as a matrix form $Ax = b$ and write down A and b.

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(b) (5 points) Solve the linear system by Cramer's rule.

7. Let $A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$.

(a) (10 points) Find the characteristic polynomial(特徵多項式), the real eigenvalues(特徵值), and the corresponding eigenvectors(特徵向量) of A.

(b) (5 points) Find an invertible matrix C and a diagonal matrix(對角矩陣) D such that $D = C^{-1}AC$.

8. (10 points) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of a linear transformation $T : V \to V$ with corresponding eigenvalues λ_1 and λ_2, respectively. Prove that, if $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent(線性獨立).

9. (10 points) Let P and Q be $n \times n$ matrices. We say P is similar to Q if there exists an invertible $n \times n$ matrix C such that $C^{-1}PC = Q$. Prove that similar square matrices have the same eigenvalues with the same algebraic multiplicities(代數重根數).

10. (10 points) Let A be an $n \times n$ matrix such that $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Show that A is an orthogonal matrix(正交矩陣).