Show all your work.
10 points each.

1. Let H be a subgroup of a group G. Prove that every coset of H in G has the same cardinality.

2. Let G_1, G_2 be cyclic groups of order m, n respectively. Find a necessary and sufficient condition of $G_1 \times G_2$ being cyclic.

3. Let $f : G \to G'$ be a surjective homomorphism of groups, H' a subgroup of G'. Let $H = f^{-1}(H')$. Prove that H' is normal in G' if and only if H is normal in G.

4. Let S_n be the group of permutations of n elements and A_n the subgroup of even permutations. Prove that A_n is a normal subgroup of index 2.

5. Prove that the kernel of a ring-homomorphism $f : R \to R'$ is an ideal of R.

6. let n be a positive integer. Show that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is a prime number.

7. Prove every ideal of the ring of integers is principal.

8. Let $R = \{a + b\sqrt{-13} : a, b \in \mathbb{Q}\}$. Prove or disprove that R is a principal ideal domain.

9. Show that $x^5 - 5x + 2003$ is irreducible in $\mathbb{Q}[x]$.

10. Suppose that $\alpha = \sqrt[5]{2}$ and $F = \mathbb{Q}(\alpha)$. Show that the degree of F over $\mathbb{Q}$ is 5.