淡江大學八十九學年度日間部轉學生招生考試試題 30

系別：數學系數學組三年級　　科目：代數

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1. Let $G = < a >$ be a cyclic group of order n.

(a) Show that for each divisor $d|n$, there is exactly one subgroup of G of order d.

(b) Find all distinct subgroups of $\mathbb{Z}_{18}$

(16 points)

2. (a) Let $GL(3, \mathbb{R})$ be the set of invertible matrices and $SL(3.\mathbb{R})$ be the set of matrices of determinant 1. Show that $GL(3, \mathbb{R})$ is a group under matrix multiplication and $SL(3, \mathbb{R})$ is a normal subgroup of $GL(3, \mathbb{R})$.

(b) Show that $GL(3, \mathbb{R})/SL(3, \mathbb{R})$ is isomorphic to $\mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$is the set of nonzero real numbers under multiplication.

(16 points)

3. Let $\phi : G \longrightarrow G'$ be a surjective homomorphism between two groups. Let N' be a normal subgroup of G'. Show that $N = \phi^{-1}(N')$ is a normal subgroup of G and G/N is isomorphic to G'/N'.

(16 points)

4. Let R be a commutative ring. Let $N = \{a \in R \,|\, a^n = 0 \text{ for some positive integer } n\}$ be the **radical** of R.

(a) Show that N is an ideal in R.

(b) Show that the radical of R/N is trivial.

(c) Find the radical of $\mathbb{Z}_{18}$.

(18 points)

5. Let $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} \,|\, a, b \in \mathbb{Q}\}$.

(a) Show that under addition and multiplication of real numbers, $\mathbb{Q}[\sqrt{3}]$ is a field .

(b) Show that $\mathbb{Q}[\sqrt{3}]$ is isomorphic to $\mathbb{Q}[x]/(x^2 - 3)$ as rings.

(16 points)

6. (a) Show that $x^3 + 2x + 1$ is an irreducible polynomial in $\mathbb{Z}_3[x]$.

(b) Show that $\mathbb{Z}_3[x]/(x^3 + 2x + 1)$ is a field of 27 elements.

(c) Let α be a root of $x^3 + 2x + 1$. Find the inverse of α.

(18 points)