淡江大學八十八學年度日間部轉學生招生考試試題

系別：數學系數學組三年級　　科目：高等代數

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Answer all questions . Show all work.

1. Let $M(\mathbf{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbf{R}\}$, where $\mathbf{R}$ is the field of real numbers.
   (a) Prove that the set T of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathbf{R}$ is a subring of $M(\mathbf{R})$.
   (b) Prove that the set I of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $b \in \mathbf{R}$ is an ideal in the ring T.
   (c) Show that every coset in T/I can be written in form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I$.

   (24%)

2. If $f : G \to H$ is a surjective homomorphism of groups and if N is a normal subgroup of G, prove that $f(N)$ is a normal subgroup of G. (10%)

3. Let $G = \mathbf{Z}_6 \times \mathbf{Z}_2$ and let N be the cyclic subgroup $\langle(1,1)\rangle$. Describe the quotient G/N. (10%)

4. Let I be an ideal in a noncommutative ring R such that $ab - ba \in I$ for all $a, b \in R$. Prove that R/I is commutative. (10%)

5. Let $\mathbf{Q}[\sqrt{2}]$ be the set of all real numbers of the form

$$r_0 + r_1\sqrt{2} + r_2(\sqrt{2})^2 + \cdots + r_n(\sqrt{2})^n,$$

   with $n \geq 0$ and $r_i \in \mathbf{Q}$, where $\mathbf{Q}$ is the field of rational numbers.
   (a) Show that $\mathbf{Q}[\sqrt{2}]$ is a subring of $\mathbf{R}$.
   (b) Show that the function $\theta : \mathbf{Q}[x] \to \mathbf{Q}[\sqrt{2}]$ defined by $\theta(f(x)) = f(\sqrt{2})$ is a surjective homomorphism, but not an isomorphism.

   (20%)

6. (a) Show that $\mathbf{Z}_2[x]/(x^3 + x + 1)$ is a field.
   (b) Show that the field $\mathbf{Z}_2[x]/(x^3 + x + 1)$ contains all three roots of $x^3 + x + 1$.

   (16%)

7. Show that $10 - 15x + 25x^2 - 7x^4$ is irreducible in $\mathbf{Q}[x]$. (10%)