淡江大學八十七學年度日間部轉學生入學考試試題

系別：數學系三年級　　　　科目：高等微積分

本試題共 / 頁

1. Let a real-valued function f be continuous at x_0 and $\lim_{x \to x_0} f'(x)$ exists. Is f differentiable at x_0? Justify your answer. (15%)

2. Let $f : [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Prove or disprove that there exists a function F such that $F'(x) = f(x)$ for all $x \in [-1,1]$. (15%)

3. Let f be continuous and decreasing on $[1,\infty)$ and let

$$a_n = f(1) + f(2) + \cdots + f(n) - \int_1^{n+1} f(x)\,dx .$$

Show that

(i) $\{a_n\}$ is an increasing sequence

(ii) $a_n \le f(1)$ for all n.

(iii) $\lim_{n\to\infty} a_n$ exists. (18%)

4. (i) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $[1+\delta,\infty)$ for each $\delta > 0$

(ii) Does $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converge uniformly on $(1,\infty)$? Justify your answer.

(iii) Let $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$, $x \in (1,\infty)$. Is $\zeta(x)$ differentiable on $(1,\infty)$? If so, give a series expansion for $\zeta'(x)$. (18%)

5. Consider the improper integral $\int_1^{\infty} \frac{\sin x}{x^p}\,dx$.

(i) Show that it converges conditionally for $0 < p \le 1$.

(ii) For what values of p, does it converge absolutely? (18%)

6. Find conditions on a point (x_0, y_0, u_0, v_0) such that there exist real-valued functions $u(x,y)$ and $v(x,y)$ which are continuously differentiable near (x_0, y_0) and satisfy the simultaneous equations

$$xu^2 + yv^2 + xy = 9$$
$$xv^2 + yu^2 - xy = 7. \qquad (16\%)$$